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A Reliable Algorithm for Obtaining Positive Solutions for Nonlinear Boundary Value Problems

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Abstract—In this paper, we propose an algorithm for solving the nonlinear two-point boundary value problem

$$u''(x) + \lambda F(x, u(x)) = 0, \qquad 0 < x < 1,$$
$$u(0) = u(1) = 0,$$

that has at least one positive solution [1–6] for λ in a compatible interval. Our method stems mainly from combining the decomposition series solution obtained by Adomian decomposition method with Padé approximates. The validity of the approach is verified through illustrative numerical examples.

Keywords—Adomian decomposition method, Nonlinear boundary value problems, Positive solutions, Padé approximates.

1. INTRODUCTION

This paper is devoted to the study of a typical example of the nonlinear boundary value problem

$$\begin{aligned} u''(x) + \lambda F(x, u(x)) &= 0, \qquad 0 < x < 1, \\ u(0) = u(1) &= 0, \end{aligned} \tag{1}$$

where $\lambda \geq 0$ and $F(x, u(x)) \in C((0,1] \times [0,\infty), [0,\infty))$ is not identically zero on any subset of $(0,1] \times [0,\infty)$.

A considerable size of research work has been invested in studying nonlinear BVPs. Several techniques, such as shooting method, finite difference method, and Green functions have been used to handle this type of problems. Research work is still carried out in this direction, and the main concern is to show formally the existence of a positive solution for problem (1) under suitable conditions of $F(x, u(x))$.

The problem has been studied by Agarwal *et al.* [2–4], Ha *et al.* [1], and O'Regan [5,6] by using various techniques. The approaches followed in these works were focused on theoretical proofs about the existence of positive solutions of boundary value problems. It was proven that

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under suitable conditions of $F(x, u(x))$, the two point boundary value problem (1) has at least one positive solution for λ belonging to a compatible interval [4]. Theorems which discuss the conditions for the existence of positive solutions of BVPs are contained in a book by Agarwal [2] and in two other books by O'Regan [5,6]. However, no numerical methods are contained in these books for solving such problems.

The present work is motivated by the desire to introduce an analytic treatment to obtain positive solutions to nonlinear BVPs. It is the hope that studying such problems will help to investigate more nonlinear applications. Our work stems mainly from Adomian decomposition method [7–14] that provides the solution in the form of a convergent series. The main advantage of Adomian decomposition method [7–14] is that it can be applied directly for all types of differential and integral equations, linear or nonlinear, homogeneous or nonhomogeneous. In most cases, five or six components of the series solution may give an insight through the behavior of the solution. However, the accuracy level can be dramatically enhanced by evaluating more terms of the series solution. The obtained series solution is converted into Padé approximates to study the behavior of the solution. Boyd [15] has formally showed that power series solution in isolation are never useful to study boundary value problems because it is only occasionally that the radius of convergence is sufficiently large to contain the boundaries. The combination of the series solution with Padé approximates was successfully used by Boyd [15], Venkatarangan [16], and Wazwaz [17,18]. Our results are new and complement existing works in the literature (see [1–6]).

2. REMARKS ON ADOMIAN DECOMPOSITION METHOD

It might seem reasonable, before launching into the main problem, to present a brief outline of few of the areas where Adomian decomposition method was used. In the last two decades, much work has been devoted in studying and using Adomian decomposition method in a variety of scientific applications that are governed by differential equations, integral equations, and integro-differential equations subject to all proper conditions. The method has been implemented in several boundary value problems with all types of boundary conditions. The validity of the method to solve differential equations subject to Dirichlet, Neumann, or Robin conditions has been justified through many works by Adomian [7–14] among many others. Recently, Deeba *et al.* [19] applied Adomian method to introduce analytical and numerical treatment of Bratu equation. Wazwaz [20,21] employed Adomian method to solve boundary value problems with Dirichlet and Neumann conditions. Recently, Wazwaz [22] has further justified the validity of using the decomposition method where mixed boundary conditions were used to obtain blow-up solutions. The results obtained in these works and in other works demonstrate the applicability of the method for all types of initial and boundary conditions.

Concerning the performance of Adomian method in scientific applications, several comparative studies were carried out between Adomian's method and other existing methods, and all results of these investigations showed that Adomian's method is effective, strong, and practical. Bellomo *et al.* [23] worked on a comparison between Adomian's method and the perturbation techniques to show that Adomian's method provides improvements over perturbation techniques and minimizes the volume of computational work. In [23], the study was conducted by analytical discussion supported by numerical examples to demonstrate the power of the method. Rach [24] conducted a constructive study, supported by useful examples, between Adomian's method and Picard's iterations. It was shown by Rach [24] that the similarity between Adomian's method and Picard's method is purely superficial. Wazwaz [25,26] carried out a useful study to compare Adomian's method with Taylor series solution method to show the improvements presented by Adomian decomposition method concerning the calculation size. Another comparative study between Adomian's method and the shooting method was conducted by Deeba *et al.* [19] by carefully investigating Bratu equation. The obtained results in [19] showed that the numerical errors obtained by Adomian's method is less than that obtained by the shooting method. Hon [27] studied Adomian's method and Green's functions method and carried out a comparison study between the performance of the methods. Working on Thomas-Fermi equation, Hon [27] showed that Green's function is not always easy to find and employing Green's functions method require a huge size of calculations compared to Adomian's method. Recently, a useful comparison between Adomian method and the variational iteration method was carried out by He [28], where both methods provide the solution in a series form. It was shown in [28] that the first method approaches the problems in a direct way, whereas the latter determines the series solution through Lagrange multipliers.

Concerning systems of differential equations, linear or nonlinear, Adomian decomposition method works effectively in handling such models. Adomian [7–9,13,14] implemented his method to handle the Brusselator differential system of equations and nonlinear dynamical systems. Wazwaz [29] examined the Brusselator system of equations and generalized the work to systems of more than two equations. Systems of three nonlinear partial differential equations in three unknown functions $u(x,y,t), v(x,y,t)$, and $w(x,y,t)$ were investigated in [29] by using Adomian decomposition method, and the obtained results demonstrate strong performance of the method in handling any system of any order.

In closing these remarks, we point out that Adomian method has been used in a wide class of differential, integral, and integro-differential equations. The method has the advantage of providing analytical approximation to a wide class of linear and nonlinear equations without linearization, perturbation, or discretization methods which may result in a massive numerical work. In addition to its use in differential and integral equations, the method was applied to integro-differential equations in [7,8,20,25,30] and in many other works.

3. ANALYSIS

It is well known in the literature [7–14] that Adomian decomposition method suggests that the solution $u(x)$ be decomposed by an infinite series of components

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \tag{2}$$

and the nonlinear function $F(x, u(x))$ by an infinite series of polynomials

$$F(x, u(x)) = \sum_{n=0}^{\infty} A_n, \tag{3}$$

where A_n are the so-called Adomian polynomials that can be generated for all types of nonlinearity according to algorithms set by Adomian [7,8]. As will be discussed later, the components $u_n(x)$ will be determined recurrently.

In an operator form, equation (1) can be written as

$$Lu = -\lambda F(x, u(x)), \tag{4}$$

where

$$L = \frac{\partial^2}{\partial x^2}, \tag{5}$$

and hence, the inverse operator L^{-1} is a two-fold integral operator given by

$$L^{-1}(.) = \int_0^x \int_0^x (.)\, dx\, dx. \tag{6}$$

Applying L^{-1} to both sides of (4) yields

$$u(x) = \alpha x - \lambda L^{-1}\left(F(x, u(x))\right), \tag{7}$$

where $\alpha = u_x(0)$.

Substituting the decomposition series (2) and (3) into both sides of (7) gives

$$\sum_{n=0}^{\infty} u_n(x) = \alpha x - \lambda L^{-1}\left(\sum_{n=0}^{\infty} A_n\right). \tag{8}$$

As stated before, the components $u_n(x)$ will be determined recurrently. To achieve this goal, the decomposition method introduces the recurrence relation

$$\begin{aligned} u_0(x) &= \alpha x, \\ u_{k+1}(x) &= -\lambda L^{-1}(A_k), \qquad k \geq 0. \end{aligned} \tag{9}$$

The first few Adomian polynomials A_n that represent the nonlinear term $G(u)$ are defined by

$$\begin{aligned} A_0 &= G(u_0), \\ A_1 &= u_1(x)G'(u_0), \\ A_2 &= u_2(x)G'(u_0) + \frac{1}{2!}u_1^2 G''(u_0), \\ A_3 &= u_3(x)G'(u_0) + u_1u_2G''(u_0) + \frac{1}{3!}u_1^3 G'''(u_0), \end{aligned} \tag{10}$$

and so on for other components (for more details see [7–12]).

In view of (9) and (10), we find

$$\begin{aligned} u_0(x) &= \alpha x, \\ u_1(x) &= -\lambda L^{-1}(A_0), \\ &= -\lambda L^{-1}(F(x, u_0(x))), \\ u_2(x) &= -\lambda L^{-1}(A_1), \\ &= -\lambda L^{-1}(u_1(x)F'(x, u_0(x))), \\ u_3(x) &= -\lambda L^{-1}(A_2), \\ &= -\lambda L^{-1}\left(u_2(x)F'(x, u_0(x)) + \frac{1}{2!}u_1^2 F''(x, u_0(x))\right), \\ u_4(x) &= -\lambda L^{-1}(A_3), \\ &= -\lambda L^{-1}\left(u_3(x)F'(x, u_0(x)) + u_1u_2F''(x, u_0(x)) + \frac{1}{3!}u_1^3 F'''(x, u_0(x))\right). \end{aligned} \tag{11}$$

Other components can be determined in a like manner. In most cases, five or six components of the series solution may give an insight through the behavior of the solution. However, the accuracy level can be dramatically enhanced by evaluating more components.

Once the components $u_n(x), n \geq 0$ are determined, the solution in a series form is constructed upon using (2), where the constant $\alpha = u_x(0)$ is as yet undetermined.

The n-term approximant

$$\phi_n = \sum_{k=0}^{n-1} u_k, \tag{12}$$

can be used to approximate the solution. The approximates $\phi_n, n \geq 2$ serve as approximate solutions, and therefore, satisfy the boundary conditions. Moreover, the Padé approximates $[n/m]$ serve as approximate solutions that satisfy the boundary conditions as well. Consequently, to determine the unknown constant α, we impose the boundary condition at $x = 1$ on an approximant $\phi_n, n \geq 2$, and on Padé approximates. This will lead to algebraic equation for each approximant. The resulting equations need only be solved to obtain a sequence of approximations for α. The limit of the sequence can be used as an exact value of α. Having determined the constant α, the solution in a series form follows immediately. For any BVP of exact solution, the obtained series will converge to that exact solution.

To give a clear overview of our discussion, two different linear and nonlinear illustrative examples will be examined.

4. NUMERICAL APPLICATIONS

EXAMPLE 1. We first consider the BVP

$$
\begin{gathered}
u''(x) + \pi^3 \frac{u^2}{\sin(\pi x)} = 0, \qquad 0 < x < 1, \\
u(0) = u(1) = 0.
\end{gathered} \tag{13}
$$

Applying the inverse operator L^{-1} on both sides of (13) and proceeding as before we obtain the relation

$$
\begin{aligned}
u_0(x) &= \alpha x, \\
u_{k+1}(x) &= -\pi^3 L^{-1}\left(\frac{1}{\sin(\pi x)} A_k\right), \qquad k \geq 0,
\end{aligned} \tag{14}
$$

where $\alpha = u_x(0)$ and A_k are Adomian polynomials that represent the nonlinear term u^2. Using (10) to derive Adomian polynomials for the nonlinear term u^2, and substituting the results into (14) will yield

$$
\begin{aligned}
u_0(x) &= \alpha x, \\
u_1(x) &= -\pi^3 L^{-1}\left(\frac{1}{\sin(\pi x)} A_0\right), \\
&= -\frac{1}{6}\alpha^2\pi^2x^3 - \frac{1}{120}\alpha^2\pi^4x^5 - \frac{1}{2160}\alpha^2\pi^6x^7 - \frac{31}{1088640}\alpha^2\pi^8x^9 + \cdots, \\
u_2(x) &= \pi^3 L^{-1}\left(\frac{1}{\sin(\pi x)} A_1\right), \\
&= \frac{1}{60}\alpha^3\pi^4x^5 + \frac{13}{7560}\alpha^3\pi^6x^7 + \frac{11}{77760}\alpha^3\pi^8x^9 + \cdots, \\
u_3(x) &= -\pi^3 L^{-1}\left(\frac{1}{\sin(\pi x)} A_2\right), \\
&= -\frac{11}{7560}\alpha^4\pi^6x^7 - \frac{31}{136080}\alpha^4\pi^8x^9 + \cdots, \\
u_4(x) &= -\pi^3 L^{-1}\left(\frac{1}{\sin(\pi x)} A_3\right), \\
&= \frac{1}{8505}\alpha^5\pi^8x^9 + \cdots.
\end{aligned} \tag{15}
$$

This gives the solution in a series form

$$
\begin{aligned}
u(x) = \alpha x &- \frac{1}{6}\alpha^2\pi^2x^3 + \left(-\frac{1}{120}\alpha^2 + \frac{1}{60}\alpha^3\right)\pi^4x^5 \\
&+ \left(-\frac{1}{2160}\alpha^2 - \frac{11}{7560}\alpha^4 + \frac{13}{7560}\alpha^3\right)\pi^6x^7 \\
&+ \left(-\frac{31}{1088640}\alpha^2 + \frac{1}{8505}\alpha^5 + \frac{11}{77760}\alpha^3 - \frac{31}{136080}\alpha^4\right)\pi^8x^9 + \cdots,
\end{aligned} \tag{16}
$$

where α is as yet undetermined.

To determine α, we first convert (16) to a polynomial. We then impose the boundary condition at $x = 1$ on the resulting polynomial and solve the resulting equation to find that

$$
\alpha = 0.9525364875. \tag{17}
$$

To determine a better approximation for the constant α, we substitute the boundary condition at $x = 1$ on the Padé approximant [3/6] of the resulting polynomial to obtain

$$
\alpha = 0.9754553439. \tag{18}
$$

It is clear that we can obtain a sequence of approximations for α by constructing other Padé approximates of other orders. This clearly gives

$$\alpha = 1, \tag{19}$$

in the limit.

Substituting (19) into (16) leads to the solution in a series form

$$u(x) = x - \frac{\pi^2}{3!}x^3 + \frac{\pi^4}{5!}x^5 - \frac{\pi^6}{7!}x^7 + \cdots, \tag{20}$$

and in a closed form by

$$u(x) = \frac{1}{\pi}\sin(\pi x), \tag{21}$$

a positive solution in $0 < x < 1$.

EXAMPLE 2. We next consider the BVP

$$\begin{gathered} u''(x) + 2\left(u'(x)\right)^2 + 8u(x) = 0, \qquad 0 < x < 1, \\ u(0) = u(1) = 0. \end{gathered} \tag{22}$$

Working with the inverse operator L^{-1} and proceeding as before we obtain the recurrence relation

$$\begin{aligned} u_0(x) &= \alpha x, \\ u_{k+1}(x) &= -L^{-1}\left(8u_k(x) + 2A_k\right), \qquad k \geq 0, \end{aligned} \tag{23}$$

where $\alpha = u_x(0)$ and $A_k, k \geq 0$ are Adomian polynomials that represent the nonlinear operator $(u'(x))^2$. Following [7,8], the first few Adomian polynomials are given by

$$\begin{aligned} A_0 &= (u_{0,x})^2, \\ A_1 &= 2u_{0,x}u_{1,x}, \\ A_2 &= 2u_{0,x}u_{2,x} + (u_{1,x})^2, \\ A_3 &= 2u_{0,x}u_{3,x} + 2u_{1,x}u_{2,x}, \\ A_4 &= 2u_{0,x}u_{4,x} + 2u_{1,x}u_{3,x} + (u_{2,x})^2. \end{aligned} \tag{24}$$

Substituting (24) into (23) yields

$$\begin{aligned} u_0(x) &= \alpha x, \\ u_1(x) &= -L^{-1}\left(8u_0 + 2A_0\right), \\ &= -\alpha^2 x^2 - \frac{4}{3}\alpha x^3 \\ u_2(x) &= -L^{-1}\left(8u_1 + 2A_1\right), \\ &= \frac{4}{3}\alpha^3 x^3 + 2\alpha^2 x^4 + \frac{8}{15}\alpha x^5, \\ u_3(x) &= -L^{-1}\left(8u_2 + 2A_2\right), \\ &= -2\alpha^4 x^4 - \frac{56}{15}\alpha^3 x^5 - \frac{88}{45}\alpha^2 x^6 - \frac{32}{315}\alpha x^7, \\ u_4(x) &= -L^{-1}\left(8u_3 + 2A_3\right), \\ &= \frac{16}{5}\alpha^5 x^5 + \frac{328}{45}\alpha^4 x^6 + \frac{1696}{315}\alpha^3 x^7 + \frac{344}{315}\alpha^2 x^8 + \frac{32}{2835}\alpha x^9. \end{aligned} \tag{25}$$

Other components were determined in a like manner.

This gives the solution in a series form

$$
\begin{aligned}
u(x) = {} & \alpha x - \alpha^2 x^2 + \left(-\frac{4}{3}\alpha + \frac{4}{3}\alpha^3\right) x^3 + \left(2\alpha^2 - 2\alpha^4\right) x^4 \\
& + \left(\frac{8}{15}\alpha - \frac{56}{15}\alpha^3 + \frac{16}{5}\alpha^5\right) x^5 + \left(-\frac{88}{45}\alpha^2 - \frac{16}{3}\alpha^6 + \frac{328}{45}\alpha^4\right) x^6 \\
& + \left(\frac{64}{7}\alpha^7 - \frac{32}{315}\alpha - \frac{4544}{315}\alpha^5 + \frac{1696}{315}\alpha^3\right) x^7 \\
& + \left(-\frac{1448}{105}\alpha^4 + \frac{344}{315}\alpha^2 - 16\alpha^8 + \frac{1808}{63}\alpha^6\right) x^8 + \cdots,
\end{aligned}
\tag{26}
$$

where α is as yet undetermined.

To determine α, we impose the boundary condition at $x = 1$ on three Padé approximates [2/3], [2/4], and [3/4] of the converted polynomial of (26), and solve the resulting equations to find that

$$\alpha = 1. \tag{27}$$

Substituting (27) into (26) gives the exact solution

$$u(x) = x - x^2. \tag{28}$$

It is obvious that this is a positive solution in the interval $0 < x < 1$.

5. DISCUSSION

The fundamental goal of this work has been to show that the two-point boundary value problem has at least one positive solution for λ belonging to a compatible interval. The goal has been achieved by using Adomian decomposition method and an approximation can be obtained to any desired number of terms. The Padé approximates have been used to accurately determine the remaining constant of the series solution.

The theorems that guarantee the existence of such a solution have been thoroughly discussed and introduced in [2,5,6]. Numerical experiments have not been contained therein. Two nonlinear models have been studied and the obtained results demonstrate the validity of the proposed scheme for this type of problems and gives the method a wider applicability.

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